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Generalized Bowen–Franks groups of integral matrices with the same zeta function☆

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ABSTRACT

In this paper the relation between the zeta function of an integral matrix and its generalized Bowen–Franks groups is studied. Suppose that A and B are nonnegative integral matrices whose invertible part is diagonalizable over the field of complex numbers and A and B have the same zeta function. Then there is an integer m, which depends only on the zeta function, such that, for any prime q such that $gcd(q, m) = 1$, for any $g(x) \in \mathbb{Z}[x]$ with $g(0) = 1$, the q-Sylow subgroup of the generalized Bowen–Franks group $BF_{g(x)}(A)$ and $BF_{g(x)}(B)$ are the same. In particular, if $m = 1$, then zeta function determines generalized Bowen–Franks groups.

1. Introduction

Zeta functions and Bowen–Franks groups are two simple but nevertheless important invariants in symbolic dynamics. Zeta function counts the number of periodic points. Bowen–Franks group is a complete invariant (apart from a sign) for flow equivalence of irreducible subshifts of finite type [6],

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which was generalized to the reducible case by Huang (cf. [7,8]). The subject was also treated in [1,2]. Bowen–Franks groups are studied as conjugacy invariants for $\mathbb{T}^n$-automorphisms in [14].

There is the well-known Latimer–MacDuffe theorem (cf. [12]) relating the ideal classes of orders and integral similarity classes. In [11], Laffey followed the approach of Olga Taussky–Todd and found a necessary and sufficient condition such that every integral matrix B which is similar to A over $\mathbb{Q}$ is integrally similar to A.

Motivated by the result in [11], in this paper we want to address the relationship between zeta function and generalized Bowen–Franks groups as invariants of shift equivalence of matrices over $\mathbb{Z}$. It turns out that under some conditions generalized Bowen–Franks groups are nearly determined by zeta function.

The following are the main results of this paper.

Theorem 1.1. *Suppose that A is a square integral matrix which is diagonalizable over the complex field. Write its characteristic polynomial as*

$$p_1^{e_1}(x)p_2^{e_2}(x)\cdots p_r^{e_r}(x),$$

where $p_i(x)(i=1,2,\ldots,r)$ are distinct monic irreducible polynomials over $\mathbb{Q}$. Choose a complex root θ_i for $p_i(x)$. Let $K_i=\mathbb{Q}(\theta_i), O_{K_i}$ be its ring of algebraic integers, $|O_{K_i}:\mathbb{Z}[\theta_i]|=m_i, C_i$ be the companion matrix of p_i and r_{ij} be the resultant of $p_i(x),p_j(x)(i\neq j)$. Let m be the least common multiple of $m_1,\ldots,m_r$ and $r_{ij}(i\neq j)$. Then there exists $t\in\mathbb{N}$ such that $I_t\otimes A$ is similar to $I_t\otimes C$ over $\mathbb{Z}\left[\frac{1}{m}\right]$, where

$$C=\underbrace{C_1\oplus\cdots\oplus C_1}_{e_1\ copies}\oplus\cdots\underbrace{C_r\oplus\cdots\oplus C_r}_{e_r\ copies}.$$

Theorem 1.2. *Suppose that A is a square integral matrix whose invertible part is diagonalizable over the complex field. Write its characteristic polynomial as*

$$x^l p_1^{e_1}(x)p_2^{e_2}(x)\cdots p_r^{e_r}(x),$$

where $p_i(x)$ $(i=1,2,\ldots,r)$ are distinct monic irreducible polynomials over $\mathbb{Q}$ and $p_i(0)\neq 0$. Choose a complex root θ_i for $p_i(x)$. Let $K_i=\mathbb{Q}(\theta_i), O_{K_i}$ be its ring of algebraic integers, $|O_{K_i}:\mathbb{Z}[\theta_i]|=m_i, C_i$ be the companion matrix of p_i and r_{ij} be the resultant of $p_i(x),p_j(x)$ $(i\neq j)$. Take $m=lcm\{m_i,r_{ij}|1\leqslant i,j\leqslant t,i\neq j\}$. Then $g(A)$ is equivalent to $I_l\oplus g(C)$ over $\mathbb{Z}\left[\frac{1}{m}\right]$, where

$$C=\underbrace{C_1\oplus\cdots\oplus C_1}_{e_1\ copies}\oplus\cdots\underbrace{C_r\oplus\cdots\oplus C_r}_{e_r\ copies}$$

for any $g(x)\in\mathbb{Z}[x], g(0)=1$.

Corollary 1.3. *Let $\zeta(x)=\left(\prod_{i=1}^t q_i(x)^{e_i}\right)^{-1}$ such that $p_i(x)=x^{deg(q_i(x))}q_i(\frac{1}{x})$ are distinct irreducible monic polynomials of degree d_i $(i=1,2,\ldots,t)$. Suppose that θ_i is a complex root of $p_i(x)$. Let $K_i=\mathbb{Q}(\theta_i), O_{K_i}$ be its ring of algebraic integers, $|O_{K_i}:\mathbb{Z}[\theta_i]|=m_i$ and r_{ij} be the resultant of $p_i(x)$ and $p_j(x)$ $(i\neq j)$. Take $m=lcm\{m_i,r_{ij}|1\leqslant i,j\leqslant t,i\neq j\}$. Let q be a prime such that $gcd(q,m)=1$. Then all integral matrices, whose invertible part is diagonalizable over the complex field and zeta function is $\zeta(x)$, have the same q-Sylow subgroup of generalized Bowen–Franks groups $BF_{g(x)}$ for any $g(x)\in\mathbb{Z}[x]$ such that $g(0)=\pm 1$. In particular, if $m=1$, then zeta function determines generalized generalized Bowen–Franks groups.*

The organization of this paper is as follows. In Section 2, we introduce basic definitions and notations. In Section 3 we prove some lemmas, and in Section 4 we give the proof of our main theorems. Finally, in Section 5, we use our result to discuss some examples in the case that A is an integral matrix of size two.

2. Basic definitions and notations

In this section we introduce some definitions and notations in linear algebra, symbolic dynamics or number theory, which will be used in later sections.

If R is a commutative ring with identity, then denote by R^n the set of n dimensional column vectors with coordinates in R, $M_n(R)$ the ring of square matrices of size n with entries in R, and $GL_n(R)$ the group of units of $M_n(R)$. If A and B are two square matrices, we make the following convention:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

and

$$A \otimes B = (a_{ij}B).$$

Definition 2.1. Two matrices A and B in $M_n(R)$ are said to be similar over R if there exists $Q \in GL_n(R)$ satisfying $Q^{-1}AQ = B$, and they are said to be equivalent over R if there exist $P, Q \in GL_n(R)$ satisfying $PAQ = B$.

Next we recall some basic definitions of shift equivalence of integral matrices and some related invariants.

Definition 2.2. Two integral square matrices A and B are said to be shift equivalent over $\mathbb{Z}$ if there are integral matrices S, T and $l \in \mathbb{N}$ such that

$$AT = TB, \quad SA = BS, \quad A^l = TS, \quad B^l = ST.$$

Remark 2.3. Every nonnegative integral square matrix A corresponds to a finite directed graph possibly with multiple edges and loops. From the graph a subshift of finite type X_A can be defined. A necessary condition for X_A to be conjugate to X_B is that A and B are shift equivalent over $\mathbb{Z}$ (cf. [16,13] or [9]).

Now we define three invariants for shift equivalence.

Definition 2.4. The zeta function of an integral matrix A is $\zeta_A(x) = det(I - Ax)^{-1}$.

Definition 2.5. Let A be an integral square matrix of size n. The eventual range $\mathfrak{R}_A$ of A is the subspace of $\mathbb{Q}^n$ defined by $\mathfrak{R}_A = \bigcap_{k=1}^{\infty} A^k\mathbb{Q}^n = A^n\mathbb{Q}^n$. The invertible part $A^\times$ of A is the restriction of linear transformation A to the eventual range, i.e., $A^\times : \mathfrak{R}_A \to \mathfrak{R}_A$ is defined by $A^\times(v) = Av$.

Definition 2.6. Let A be a square integral matrix of size n. Suppose $g(x) \in \mathbb{Z}[x]$ such that $g(0) = 1$. The generalized Bowen–Franks group of A associated to $g(x)$ is $BF_{g(x)}(A) = \mathbb{Z}^n/g(A)\mathbb{Z}^n$. In the particular case $g(x) = 1 - x$, we get the Bowen–Franks group of A: $BF(A) = \mathbb{Z}^n/(I - A)\mathbb{Z}^n$.

Remark 2.7. Generalized Bowen–Franks group of a matrix A over a general commutative ring R with identity is defined in [9]. If we take $R = \mathbb{Z}[t], g(x) = 1 - tx$, then we get the dimensional group $Coker(I_n - At) = \mathbb{Z}[t]^n/(I_n - At)\mathbb{Z}[t]^n$, which was complete invariants for shift equivalence of integral matrices under some condition (cf. [13] or [9]).

As a simple consequence of the theory of Smith normal form for matrices over a principle ideal domain(cf. [15]), we have $BF_{g(x)}(A) = G = \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \mathbb{Z}_{d_r} \oplus \mathbb{Z}^{n-r}$, where $d_i|d_{i+1}$ and $\mathbb{Z}_d = (\mathbb{Z}, +)/d\mathbb{Z}$. It can also be written as the direct sum of its p-Sylow subgroups G_p and some copies of $\mathbb{Z}$, where G_p is trivial for all but finitely many primes p.

The following are another two definitions which will be used.

Definition 2.8. An integral square matrix A of size n is said to be derogatory modulo a prime p if the degree of the minimal polynomial of A mod p as a matrix over the finite field $\mathbb{Z}_p = (\mathbb{Z}, +, \cdot)/p\mathbb{Z}$ is less than n. Otherwise A is said to be nonderogatory modulo p.

Definition 2.9. (cf. [4] for more general definitions) Let K be a number field of extension degree n over $\mathbb{Q}$. Then an order O of K is a subring with identity of the ring O_K of algebraic integers of K with n generators over $\mathbb{Z}$. If θ is an algebraic integer in K, then $\mathbb{Z}[\theta]$ is an order of $K = \mathbb{Q}[\theta]$. The finite index of O as a subgroup in O_K is denoted by $|O_K : O|$.

3. Some lemmas

The idea of almost all of the proofs of the following lemmas is from the corresponding results in Lectures 2 and 3 of [11]. However, for the convenience of the readers, all the proofs will be included.

Lemma 3.1. *(1) Any integral square matrix is similar over $\mathbb{Z}$ to an upper triangular matrix of the following form:*

$$\begin{pmatrix} A_{11} & * & * & * & * \\ & A_{22} & * & * & * \\ & & \ddots & \vdots & \vdots \\ & & & A_{t-1\,t-1} & * \\ & & & & A_{tt} \end{pmatrix},$$

where A_{ii} have irreducible characteristic polynomials.

(2) Suppose that an integral square matrix A is diagonalizable over complex field. Then A is similar over $\mathbb{Z}$ to an upper triangular matrix of the following form:

$$\begin{pmatrix} A_{11} & * & * & * & * \\ & A_{22} & * & * & * \\ & & \ddots & \vdots & \vdots \\ & & & A_{t-1\,t-1} & * \\ & & & & A_{tt} \end{pmatrix},$$

where A_{ii} have irreducible minimal polynomials.

Proof. Let $p(x)$ be an irreducible factor of the characteristic polynomial of an integral matrix A of size n, and θ be a complex root of $p(x)$. Let $K = \mathbb{Q}(\theta)$. Denote the ring of algebraic integers in K by O_K. As a group it is a free abelian group of rank $r = [K : \mathbb{Q}] = \deg p(x)$. So it has an integral basis $\omega_1, \omega_2, \ldots, \omega_r$, which is also a vector space basis for K over $\mathbb{Q}$.

Let X be a column eigenvector of A corresponding to θ with the entries of X in O_K. Then $X = C\omega$ for some $n \times k$ integral matrix C where $\omega = (\omega_1, \ldots, \omega_k)^T$. Also $\theta\omega = B\omega$ for some $k \times k$ integer matrix B.

Now $AX = \theta X$ implies $AC\omega = \theta C\omega = C\theta\omega = CB\omega$, so $(AC - CB)\omega = 0$. We can write $C = USV$ where $U \in GL_n(\mathbb{Z}), V \in GL_k(\mathbb{Z})$ and S is the Smith normal form of C.

Write

$$U^{-1}AU = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is $k \times k$. Then we have

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} V\omega = \begin{pmatrix} S_1 \\ 0 \end{pmatrix} VB\omega = \theta \begin{pmatrix} S_1 \\ 0 \end{pmatrix} V\omega,$$

where S_1 is $k \times k$ and is the top part of the Smith normal form S. Hence $A_{11}z = \theta z$, where $\binom{z}{0} = \binom{S_1}{0} V\omega$, and $A_{21} = 0$.

Thus A_{11} has characteristic polynomial $p(x)$ and

$$U^{-1}AU = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

Proceeding in this way, we find that there is $Q \in GL(n, \mathbb{Z})$ with

$$Q^{-1}AQ = \begin{pmatrix} A_{11} & * & * & * & * \\ & A_{22} & * & * & * \\ & & \ddots & \vdots & \vdots \\ & & & A_{r-1,r-1} & * \\ & & & & A_{rr} \end{pmatrix}$$

for some $r \geqslant 1$, where the diagonal blocks A_{ii} have irreducible characteristic polynomials. The proof of (1) is complete. It is easy to deduce (2) from (1). $\square$

Lemma 3.2. *Let $p(x) \in \mathbb{Z}[x]$ be a monic irreducible polynomial of degree n, θ be a complex root of $f(x)$. Let $K = \mathbb{Q}(\theta)$ and O_K be the ring of algebraic integers of K, m be the index $|O_K : \mathbb{Z}[\theta]|$ and p be a rational prime number.*

(1) *If $gcd(p, m) = 1$, then any matrix $A \in M_n(\mathbb{Z})$ with characteristic polynomial $f(x)$ is nondegroatory mod p;*

(2) *If $p|m$, then there exists $A \in M_n(\mathbb{Z})$ with characteristic polynomial $f(x)$ such that A is derogatory modulo p.*

Proof. (1) Suppose A has characteristic polynomial $f(x)$ and that A mod p is derogatory. Then there exist integers $a_0, a_1, \ldots, a_{n-1}$ not all divisible by p such that $a_0I + a_1A + \cdots + a_{n-1}A^{n-1} = pB$ where B is a matrix of integers. But then $\beta = \frac{a_0+a_1\theta+\cdots+a_{n-1}\theta^{n-1}}{p}$ is an algebraic integer and it is not in $\mathbb{Z}[\theta]$ since $1, \theta, \ldots, \theta^{n-1}$ are linearly independent. If $gcd(p, m) = 1$, there exists $u, v \in \mathbb{Z}$ such that $pu + mv = 1$. Thus $\beta = pu\beta + mv\beta \in \mathbb{Z}[\theta]$. A contradiction arises.

(2) Suppose $p \| O_K : \mathbb{Z}[\theta]|$. Let $\beta = \frac{b_0+b_1\theta+\cdots+b_{n-1}\theta^{n-1}}{p}$ be an element of O_K not in $\mathbb{Z}[\theta]$, where $b_0, \ldots, b_{n-1} \in \mathbb{Z}$ and p is some prime. Let $\omega_1, \ldots, \omega_n$ be an integral basis for O_K and let A be the matrix describing multiplication by θ on this basis. Then $\frac{b_0I+b_1A+\cdots+b_{n-1}A^{n-1}}{p}$ is the matrix describing multiplication by β on this basis. So it has integer entries and A mod p is derogatory. $\square$

Remark 3.3. It follows from the lemma above that if $O_K = \mathbb{Z}[\theta]$, then A mod p is nonderogatory for any prime p. In general, $O_K \neq \mathbb{Z}[\theta]$ and A is nonderogatory for all but finitely many primes.

We next consider the similarity of direct sums

$$\underbrace{A \oplus \cdots \oplus A}_{k \text{ copies}} = I_k \otimes A$$

and

$$\underbrace{B \oplus \cdots \oplus B}_{k \text{ copies}} = I_k \otimes B$$

of given integral matrices A and B of size n.

Lemma 3.4. *Suppose that $A \in M_n(\mathbb{Z})$ have an irreducible characteristic polynomial $p(x) \in \mathbb{Z}[x]$. Let θ be a complex root of $f(x)$. Let $K = \mathbb{Q}(\theta)$ and O_K the ring of algebraic integers of K, m be the index $|O_K : \mathbb{Z}[\theta]|$. Then there exists $k \in \mathbb{N}$ such that $I_k \otimes A$ is similar to $I_k \otimes C$ over $\mathbb{Z}\left[\frac{1}{m}\right]$, where C is the companion matrix of $p(x)$.*

Proof. Consider the equation $CX = XA$ where $X = (x_{ij})$. Using the equations obtained by comparing the first $n-1$ rows, we can write (x_{ij}) $(i \geqslant 2)$ in terms of $x_{11}, \ldots, x_{1n}$. Having done this substitution, consider $\det(X)$. This is a homogeneous form of degree n in $x_{11}, \ldots, x_{1n}$ and it has integer coefficients and the hypotheses guarantee that the greatest common divisor d of the coefficients is a factor of some power of m by Lemma 3.2. By a theorem of Dade in [5], this form represents some power of m on the ring of all algebraic integers. Let F be the finite extension of $\mathbb{Q}$ generated by algebraic integers $x_{11}, \ldots, x_{1n}$ chosen with $\det(X) = d$. Let $\omega_1, \ldots, \omega_k$ be a basis of O_F and let $X_{11}, \ldots, X_{1n}$ be the matrices representing multiplication by $x_{11}, \ldots, x_{1n}$ on this basis, and let X be the corresponding $kn \times kn$ matrix. Replacing the entries y of A and C by yI_k (so A and C are replaced by $A \otimes I_k$ and $C \otimes I_k$), we find an integral matrix X with $\det(X) = d$ and $X^{-1}(A \otimes I_k)X = C \otimes I_k$ and using a permutation similarity we get that $I_k \otimes A$ is similar to $I_k \otimes C$ over $\mathbb{Z}\left[\frac{1}{m}\right]$. □

An example is given in [11] that $A \oplus A$ is integrally similar to $A^T \oplus A^T$, where A is the transpose of A, but A is not integrally similar to A^T. However, we have the following simple lemma, which is essential to the proof of our main result.

Lemma 3.5. *If there exists $k, m \in \mathbb{N}$ such that $I_k \otimes A$ is similar to $I_k \otimes C$ over $\mathbb{Z}\left[\frac{1}{m}\right]$, then A is equivalent to C over $\mathbb{Z}\left[\frac{1}{m}\right]$.*

Proof. Note that if there are $U, V \in GL_n\left(\mathbb{Z}\left[\frac{1}{m}\right]\right)$ such that

$$UAV = D = diag(s_1, \ldots, s_n)$$

then $(U \otimes I_k)(A \otimes I_k)(V \otimes I_k) = D \otimes I_k$. So the smith normal form of $A \otimes I_k$ uniquely determines that of A. The result is proved. □

Lemma 3.6. *Suppose that $A \in M_n(\mathbb{Z})$ have an irreducible minimal polynomial $p(x)$ of degree d. Let θ be a complex root of $p(x)$. Let $K = \mathbb{Q}(\theta)$ and O_K the ring of algebraic integers of K, m be the index $|O_K : \mathbb{Z}[\theta]|$ and C be the companion matrix of $p(x)$. Then there exists $t \in \mathbb{N}$ such that $I_t \otimes A$ is similar to $I_k \otimes C$ over $\mathbb{Z}\left[\frac{1}{m}\right]$, where $k = \frac{tn}{d}$.*

Proof. By Lemmas 3.1 and 3.4, there exists $t \in \mathbb{N}$ such that

$$\widetilde{A} = A \otimes I_t = \begin{pmatrix} C_{11} & C_{12} & \cdots & \cdots & C_{1k} \\ & C_{22} & \ddots & \ddots & C_{2k} \\ & & \ddots & \ddots & \vdots \\ & & & C_{k-1,k-1} & C_{k-1,k} \\ & & & & C_{k,k} \end{pmatrix},$$

where $k = \frac{tn}{d}$, $C_{ii} = C$. Since $\widetilde{A}$ has minimal polynomial $p(x)$, there exists a matrix T with integer entries such that $T^{-1}\widetilde{A}T = I_k \otimes C = C \oplus \cdots \oplus C$. Write $T = (T_{ij})$ in block form. Comparing the last block row of the matrices in

$$\widetilde{A}T = T\begin{pmatrix} C & & \\ & \ddots & \\ & & C \end{pmatrix},$$

we get

$$CT_{ki} = T_{ki}C$$

for all i. Comparing the second last row gives

$$CT_{k-1\,i} + C_{k-1\,k}T_{ki} = T_{k-1\,i}C.$$

Since T_{ki} commutes with C, it is either 0 or nonsingular and thus for some j, $\det T_{kj} \neq 0$. The last equation with $i = j$ now yields

$$C_{k-1\,k} = (T_{k-1\,j}C - CT_{k-1\,j})T_{kj}^{-1} = (T_{k-1\,j}T_{kj}^{-1})C - C(T_{k-1\,j}T_{kj}^{-1})$$

Let $Y = T_{k-1\,j}T_{kj}^{-1}$. Then $C_{k-1\,k} = YC - CY$. Hence $C_{k-1\,k} \in [M_d(\mathbb{Q}), C] \cap M_d(\mathbb{Z})$.

Let s be the smallest positive integer with

$$sC_{k-1\,k} \in [M_d(\mathbb{Z}), C].$$

Suppose that $s > 1$. Let π be a prime dividing s. If

$$sC_{k-1\,k} = [W, C],$$

with $W \in M_d(\mathbb{Z})$, then, read mod π, $[W, C] \equiv 0$.

But C, being a companion matrix, is nonderogatory modulo every prime. So, as a matrix over $\mathbb{Z}_\pi$, $W = f(C)$ for some polynomial $f(x) \in \mathbb{Z}[x]$. But in terms of integral matrices, this says that $W - f(C) = \pi V$ where V is a matrix of integers. But then $sC_{k-1\,k} = \pi[V, C]$ and $\frac{s}{\pi}C_{k-1\,k} = [V, C]$. This contradicts the minimality of s. So $s = 1$, which means that $C_{k-1k} \in [M_d(\mathbb{Z}), C]$.

Writing $C_{k-1\,k} = [V, C]$ with V an integral matrix, we perform a similarity on A using

$$I \oplus \ldots \oplus I \oplus \begin{pmatrix} I & V \\ 0 & I \end{pmatrix}.$$

The effect is to replace C_{k-1k} in A by 0.

This step provides the starting point for an inductive proof. The proof is complete. $\square$

4. Proofs of the main results

Let us begin to prove Theorem 1.1.

Proof. Suppose that A is of size n. Since A is diagnalizable over complex field, by Lemma 3.1, we can assume that

$$A = \begin{pmatrix} A_{11} & * & * & * \\ 0 & A_{22} & * & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{rr} \end{pmatrix},$$

where A_{ii} has minimal polynomial $p_i(x)$ $(i = 1, 2, \ldots, r)$. Consider a similarity of A by

$$\begin{pmatrix} I & & & & & \\ & \ddots & & & & \\ & & I & & X & \\ & & & \ddots & & \\ & & & & I & \\ & & & & & I \end{pmatrix},$$

where X is a matrix of integers placed in the (i,j) block position. The effect is to replace A_{ij} by $A_{ij} + (A_{ii}X - XA_{jj})$. Suppose A_{ii} is $k \times k$ and A_{jj} is $l \times l$. The map $M_{k,l}(Z) \to M_{k,l}(Z) : X \mapsto A_{ii}X - XA_{jj}$ is an additive linear mapping with matrix $A_{ii} \otimes I_l - I_k \otimes A_{jj}^T$, which is invertible over $\mathbb{Z}\left[\frac{1}{r_{ij}}\right]$, here r_{ij} is the resultant of $p_i(x), p_j(x)$. So the map is invertible over $\mathbb{Z}\left[\frac{1}{r(i,j)}\right]$.

Using this process we can replace the blocks $A_{ij} (i \neq j)$ by 0 (note that doing the last column first etc does not interfere with already created zeros). The result follows from Lemma 3.6. $\square$

The following is the proof of Theorem 1.2.

Proof. Suppose that A is of size n. By Lemma 3.1, we can assume that

$$A = \begin{pmatrix} N & D \\ 0 & A_1 \end{pmatrix},$$

where N is a nilpotent matrix of size l, A_1 has characteristic polynomial $\prod_{i=1}^{r} p_i(x)^{e_i}$ and is diagonalizable over $\mathbb{C}$.

Then we have H such that

$$g(A) = \begin{pmatrix} g(N) & H \\ 0 & g(A_1) \end{pmatrix}.$$

Note that $g(0) = 1$, we have $det(g(N)) = 1$ and thus $g(N) \in GL_l\left(\mathbb{Z}\left[\frac{1}{m}\right]\right)$. Take

$$P = \begin{pmatrix} g(N)^{-1} & -g(N)^{-1}H \\ 0 & I_{n-l} \end{pmatrix}.$$

Then

$$g(A)P = I_l \bigoplus g(A_1).$$

By Theorem 1.1 there exist $t \in \mathbb{N}$ and $Q \in GL_{(n-l)t}\left(\mathbb{Z}\left[\frac{1}{m}\right]\right)$ such that $Q^{-1}(I_t \bigotimes A_1)Q = I_t \bigoplus C$, where C is defined as in Theorem 1.1 for A_1. So we have

$$Q^{-1}\left(I_t \bigotimes g(A_1)\right)Q = I_t \bigotimes g(C).$$

The result follows from Lemma 3.5. □

Corollary 1.3 follows easily from Theorem 1.2. So the proof is omitted here.

Remark 4.1. (1) If the A is a symmetric integral square matrix, eg., the graph associated to A is a symmetric directed graph (un-oriented graph), then A is diagonalizable over $\mathbb{R}$ and thus our result can be applied.

(2) Suppose that A and B are nonnegative integral square matrices. By Theorem 2.4.6 in [10], a necessary condition for them to be shift equivalent over $\mathbb{Z}$ is that their invertible parts are similar over $\mathbb{C}$. In this case, if one of them is diagonalizable over $\mathbb{C}$, so is the other.

5. Some examples

To illustrate our results, in this section we will consider the generalized Bowen–Franks groups for two by two integral matrices.

Suppose that $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be an integral matrix. Then

$$det(xI_2 - A) = x^2 - (\alpha + \delta)x + \alpha\delta - \beta\gamma = x^2 - bx + c$$

and

$$\frac{1}{det(I_2 - xA)} = \frac{1}{1 - bx + cx^2}$$

are the characteristic polynomial and zeta function of A respectively.

Suppose that $D = b^2 - 4c = f^2 D_0 \neq 0$, where $f > 0$, and D_0 is a square-free integer. Let

$$\theta = \frac{b + \sqrt{D}}{2} = \frac{b + f\sqrt{D_0}}{2} = \frac{b - f}{2} + f\frac{1 + \sqrt{D_0}}{2}.$$

Since $\mathbb{Z}[\omega]$ is the ring of integers of $\mathbb{Q}[\theta]$, where

$$\omega = \begin{cases} \frac{1+\sqrt{D_0}}{2} & \text{if } D_0 \equiv 1 \ (mod\, 4) \\ \sqrt{D_0} & \text{otherwise} \end{cases}$$

(cf. [3]), the index $|\mathbb{Z}[\omega] : \mathbb{Z}[\theta]|$ is

$$m = \begin{cases} f & \text{if } D_0 \equiv 1\ (mod\, 4) \\ \frac{f}{2} & \text{otherwise.} \end{cases}$$

If $m = 1$, then zeta function determines the generalized Bowen–Franks groups by Corollary 1.3. The following is an example in which $m > 1$. Let

$$A = \begin{pmatrix} 3 & 4 \\ 4 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 2 & 17 \\ 1 & 2 \end{pmatrix}.$$

Then they have the same zeta function $\frac{1}{1-4x-13x^2}$. Since the ring $\mathbb{Z}[2+\sqrt{17}] = \mathbb{Z}[1+\sqrt{17}]$ has index 2 as a subgroup of the ring of algebraic integers $\mathbb{Z}\left[\frac{1+\sqrt{17}}{2}\right]$, we have $m = 2$. So only 2-torsion part of the generalized Bowen–Franks groups for the same $g(x)(g(0) = 1)$ might be different. In fact, $BF(A) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and $BF(B) = \mathbb{Z}/8\mathbb{Z}$. This example is from Example 2.2.4 in Kitchens' book [9].

Now we consider some examples in which the characteristic polynomial is reducible. First let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}.$$

Then they have the same characteristic polynomial $x^2 - 2x - 3 = (x-3)(x+1)$. By Corollary1.3, we have $m = 4$, and thus only 2-Sylow subgroup of the generalized Bowen–Franks groups for the same $g(x)(g(0) = 1)$ might be different. In fact, $BF(A) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $BF(B) = \mathbb{Z}/4\mathbb{Z}$. Another example is in Exercise 7.4.12 in [13]:

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 6 & 8 \\ 1 & 4 \end{pmatrix}.$$

Then they have the same Zeta functions and Bowen–Franks group. However, in this case $m = 6$. By Corollary 1.3, only 2-Sylow or 3-Sylow subgroup of the generalized Bowen–Franks groups might be different. In fact, taking $g(x) = 1 + x$, we have different generalized Bowen–Franks groups:

$$BF_{1+x}(A) = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$$

and

$$BF_{1+x}(B) = \mathbb{Z}/27\mathbb{Z}.$$

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